Chaos in Classical Systems

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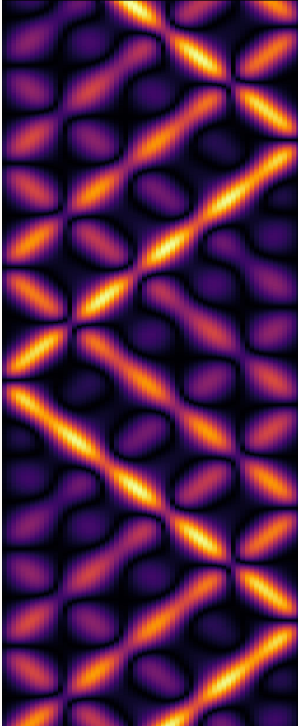
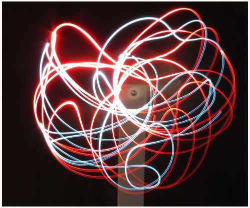


- Introduction
- Probing Classical Chaos via the AGP
- Observable Diffusion
- Numerical Example: Fermi–Pasta–Ulam–Tsingou (FPUT) Systems
- Beyond the AGP
- Conclusions

Introduction

- **What does chaos mean?**
  Unpredictability, complex dynamics, randomness.
- Arises from deterministic dynamics, and yet information of the initial state is quickly lost.



Bowie & Cotton, 2016

- **Lyapunov exponents (λ):** Sensitivity to initial conditions.

$$\delta x(t) \approx e^{\lambda t} \delta x(0).$$

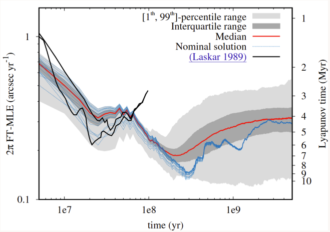
- Time-scale of exponential separation: Lyapunov time.

$$t_{\text{Lyapunov}} = 1/\lambda.$$

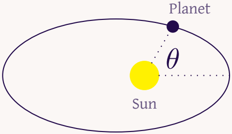
Major issues with the Lyapunov exponent!

Lyapunov time does not actually represent chaos time-scales.

The Lyaponov time of the inner solar system is $\sim$ 5 Myrs. The solar system is already more than 4.5 Gyrs old!



Mogavero & Laskar, 2021



The Lyapunov time corresponds to dephasing of the orbit. Eccentricities change over exponentially longer times.

Lyaponov time $\neq$ mixing time.

Introduction

Notion of exponential separation does not generalize to quantum systems.

$$\rho(t=0) = \sum_{m,n} c_{mn} |m\rangle \langle n| \implies \rho(t \to \infty) \approx \sum_{m} c_{mm} |m\rangle \langle m| .$$

No unified picture of chaos.

Proposed definition of both classical and quantum chaos: Sensitivity to adiabatic changes.[1]

$$H(\lambda) \to H(\lambda + \delta\lambda)$$

How the system reacts to adiabatic perturbations depends on the nature of chaos.

Deformations are captured by the ***Adiabatic Gauge Potential (AGP).***

[1] Lim, C., Matirko, K., Kim, H., Polkovnikov, A., & Flynn, M.. (2024). Defining classical and quantum chaos through adiabatic transformations.

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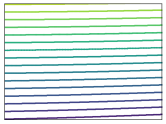
Classical Adiabatic Gauge Potential

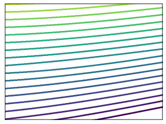
Adiabatic Gauge Potential (AGP): Function in the phase space, $\mathcal{A}_\lambda(x,p)$. Generator of adiabatic deformations.

$$\frac{\partial x}{\partial \lambda} = \{x, \mathcal{A}_\lambda\},\ \frac{\partial p}{\partial \lambda} = \{p, \mathcal{A}_\lambda\}.$$



$H(\lambda)$

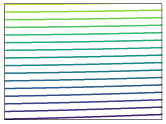
$\xrightarrow{\mathcal{A}_\lambda}$



$H(\lambda + \delta\lambda)$

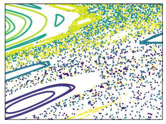
Small Deformations of Trajectories

Integrable



$H(\lambda)$

$\xrightarrow{\mathcal{A}_\lambda}$



$H(\lambda + \delta\lambda)$

Large Deformations of Trajectories

Chaotic

Classical Adiabatic Gauge Potential

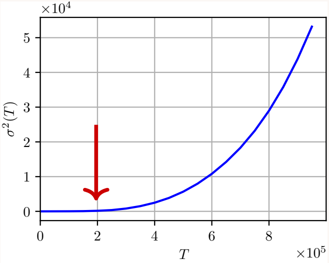
Claim: The variance of the AGP over a trajectory in phase space is a measure of chaos on that trajectory.

AGP variance over a trajectory $(x(t), p(t))$:

$$\sigma^2(T) = \frac{1}{T}\int_0^T dt\ \mathcal{A}_\lambda^2(x(t), p(t)) - \left(\frac{1}{T}\int_0^T dt\ \mathcal{A}_\lambda(x(t), p(t))\right)^2$$

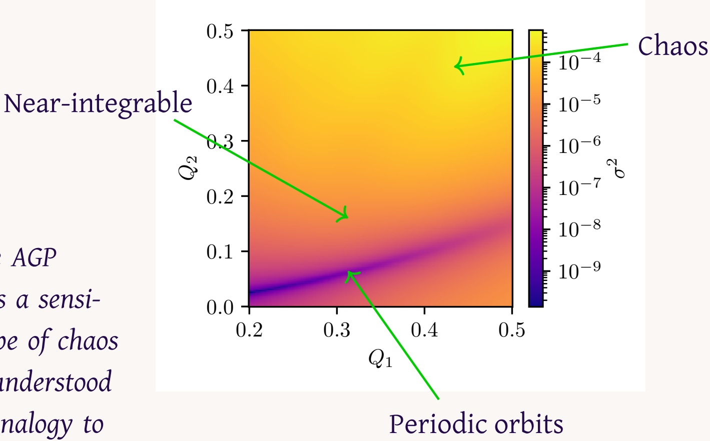
AGP variance is computed over a time window $[0, T] \rightarrow$ captures chaotic behavior within the window.

Onset of chaos is detected when AGP variance begins to grow



Classical Adiabatic Gauge Potential

Example: Phase space of an α-FPUT system. AGP variance over trajectories is plotted as a function of the initial conditions.



Why the AGP serves as a sensitive probe of chaos can be understood via an analogy to diffusion.

An Analogy to Diffusion

AGP along a trajectory evolves as:

$$\underbrace{\frac{d\mathcal{A}_\lambda}{dt}}_{\text{Position}} = \underbrace{-\,\partial_\lambda H(t) + \overline{\partial_\lambda H}}_{\text{Velocity}}.$$

Consider an ensemble of initial conditions, $\rho(x, p)$. If the AGP along trajectories in this ensemble is identified with positions, $\partial_\lambda H$ serves as the velocity.

Mean squared displacement is analogous to $\langle \sigma^2(T) \rangle$.

Fluctuation-Dissipation Relation (FDR):

$$\langle \sigma^2(T) \rangle = \frac{T}{6} \int_{-T}^{T} dt \left(1 - \frac{|t|}{T}\right)^3 \langle \partial_\lambda H(t) \partial_\lambda H(0) \rangle_c .$$

An Analogy to Diffusion

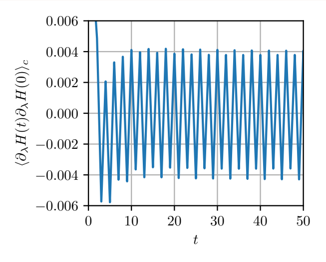
The FDR allows us to probe large-time correlations via the mean variance of the AGP.

$$\langle \sigma^2(T \to \infty) \rangle \leftrightarrow \langle \partial_\lambda H(t \to \infty) \partial_\lambda H(0) \rangle_c.$$

Integrable Systems: Quasi-periodic correlations.

$$\langle \sigma^2(T \to \infty) \rangle = \text{constant}.$$

No diffusion



An Analogy to Diffusion

Non-integrable Systems: Correlations may appear quasi-periodic over short time-scales, but eventually decay.

If the decay is slower than $\mathcal{O}(1/t)$:

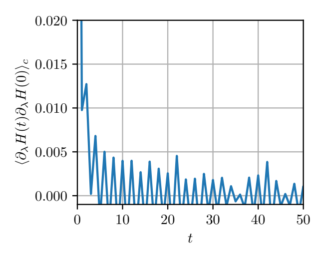
$$\langle \sigma^2(T \to \infty) \rangle \sim T^\gamma,\ 1 < \gamma < 2.$$

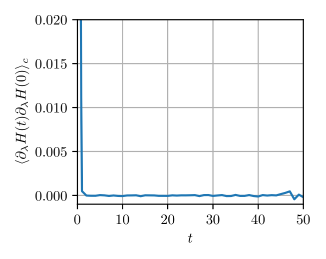
Anomalous diffusion

If the decay is faster than $\mathcal{O}(1/t)$:

$$\langle \sigma^2(T \to \infty) \rangle \sim T.$$

Normal diffusion





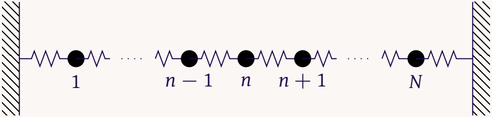
An Analogy to Diffusion

Recipe for identifying regimes of chaos:

$$\langle \sigma^2 (T \to \infty) \rangle \sim \begin{cases} \text{constant}, & \text{Integrable} \\ T^\gamma \; (1 < \gamma \leq 2), & \text{Weak Chaos (Maximal)} \\ T, & \text{Strong Chaos.} \end{cases}$$

FPUT Systems

1D chain of oscillators with non-linear interactions.



Hamiltonian in Fourier mode space:

$$H = \sum_{k=1}^{N} \frac{P_k^2 + \omega_k^2 Q_k^2}{2} + \text{Non-linear terms.}$$

Without non-linearity, $E_k = \frac{P_k^2 + \omega_k^2 Q_k^2}{2}$ is conserved. Nonlinear part:

α-FPUT

$$\frac{\alpha}{3} \sum_{i,j,k} A_{ijk} Q_i Q_j Q_k$$

Cubic

β-FPUT

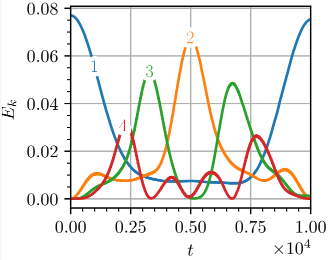
$$\frac{\beta}{4} \sum_{i,j,k,l} B_{ijkl} Q_i Q_j Q_k Q_l$$

Quartic

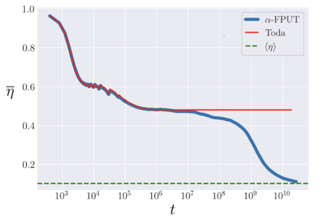
FPUT Systems

One of the first systems to be numerically studied. The system was expected to show equipartition.

FPUT Paradox: If the system is initialized in a single mode, almost all the energy returns periodically.



Fermi, Pasta, Ulam, & Tsingou, 1955



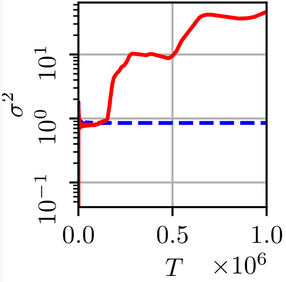
Reiss & Campbell, 2023

FPUT systems do eventually thermalize, but over exponentially long times. Go through a long-lived, non-thermal, ***metastable*** phase.

Would like to understand the FPUT problem through the lens of the AGP.

AGP variance along individual trajectories:

Blue: Lower energy, $\sigma^2(T)$ plateaus, trajectory remains in the metastable state.
Red: Higher energy, $\sigma^2(T)$ grows, metastable state breaks down.

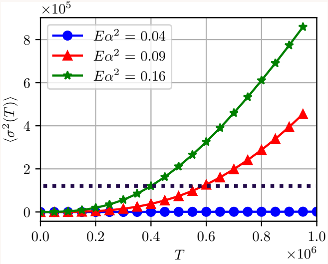


α-FPUT

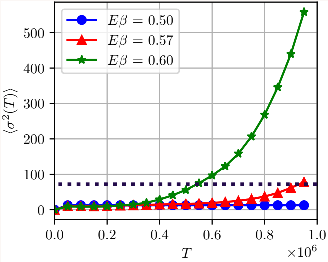
To see diffusive behavior, need to choose an ensemble of trajectories.
We consider trajectories initialized in the first mode, with uniformly distributed phase, and study $\langle \sigma^2(T) \rangle$.

Chaos in FPUT Systems

Transition to weak chaos is observed in both α-FPUT and β-FPUT systems.



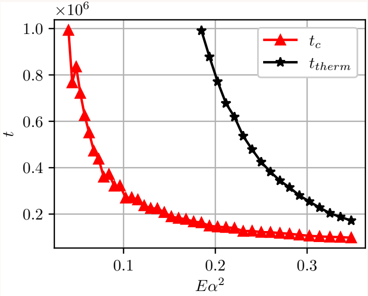
α-FPUT



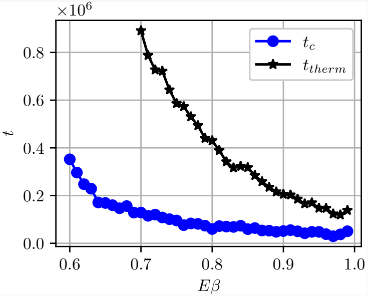
β-FPUT

- Mean AGP variance is initially constant, but grows non-linearly after the breakdown of the metastable state.
- The growth rate increases with non-linearity.
- "Onset time of chaos" measured w.r.t. a threshold.

Chaos is observed much earlier than thermalization.



α-FPUT

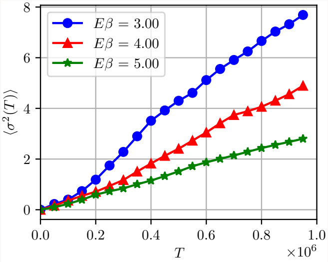


β-FPUT

The trajectory stays close to an integrable regime while in the metastable state. During the breakdown of the metastable state, the trajectory becomes weakly chaotic and starts exploring the rest of the phase space. Trajectory becomes ergodic when it thermalizes.

Chaos in FPUT Systems

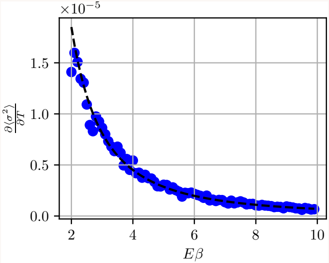
Transition to strong chaos is observed after thermalization.



β-FPUT

- AGP variance grows linearly.
- Growth rate slows down with increasing non-linearity.

Growth rate $\propto 1/\beta^2$. The Lyapunov time has the same scaling!



Numerically demonstrated how the AGP can be used as a probe of chaos in the FPUT system.

$$\langle \sigma^2 (T \to \infty) \rangle \sim \begin{cases} \text{constant}, & \text{Integrable} \ \textcolor{red}{\longrightarrow \text{Metastable}} \\ T^\gamma \ (1 < \gamma \leq 2), & \text{Weak Chaos} \ \textcolor{red}{\longrightarrow \text{Before thermalization}} \\ T, & \text{Strong Chaos.} \ \textcolor{red}{\longrightarrow \text{After thermalization}} \end{cases}$$

How do we generalize this to other systems?

Not clear how to define AGP in non-Hamiltonian systems (discrete maps, dissipative systems). Need a more general approach.

Beyond the AGP

Evolution of a trajectory in phase space:

Integrable regime: A trajectory starts near an integrable regime, and it remains localized for some time.

Weak regime: The trajectory starts exploriong other parts of the phase space.

Strong regime: When it visits all of the accessible phase space, it becomes ergodic.

Ergodic Hypothesis: When a trajectory is ergodic, its time average converges to its phase space average:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, O(t) = \langle O \rangle,$$

Claim: *How* the time average converges to the phase space average depends on how the observable diffuses, allowing us to probe chaos.

Beyond the AGP

Observable drift:

$$\Delta(T) = \int_0^T dt\, O(t) - T\overline{O}.$$

Measures the separation between the time integral of an observable and its large-time average. AGP is the drift of the perturbation!

Mean drift variance is a measure of chaos.

$$\sigma^2(T) = \frac{1}{T}\int_0^T dt\, \Delta^2(T) - \left(\frac{1}{T}\int_0^T dt\, \Delta(T)\right)^2.$$

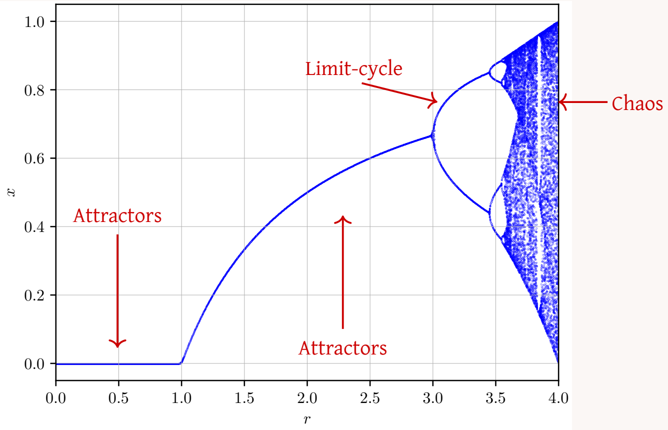
$\langle \sigma^2(T \to \infty) \rangle \sim T^{\gamma}$

$-1 \leq \gamma < 0$	Fixed-Point Attractor
$\gamma = 0$	Limit-Cycle / Integrable
$1 < \gamma \leq 2$	Weak Chaos
$\gamma = 1$	Strong Chaos

Example: Logistic Map

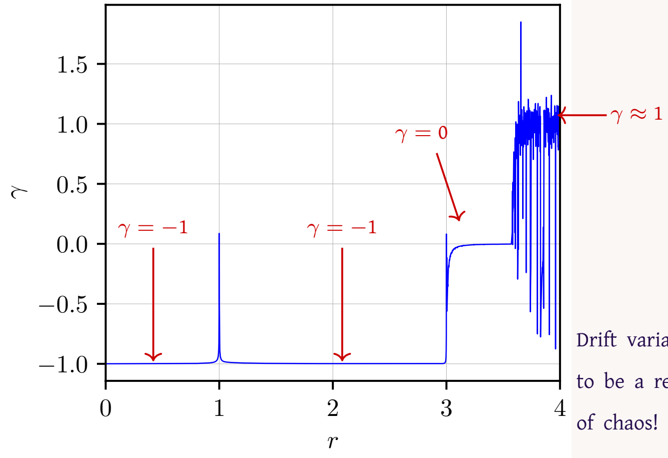
Logistic Map: $x_{n+1} = rx_n(1 - x_n)$, $0 < r < 4$.

Simple, 1D system, with interesting chaotic behavior.



Example: Logistic Map

We study the drift of x_n: $\Delta_N = \sum_{n=0}^{N} x_n - N\bar{x}$, variance: $\langle \sigma_N^2 \rangle \sim N^{\gamma}$.



Drift variance appears to be a reliable probe of chaos!

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Conclusions

- Demonstrated how AGP variance is related to observable diffusion and can probe chaos in FPUT systems.
- Can be generalized to other non-Hamiltonian dynamical systems, unifying our understanding of chaos.

Further Questions

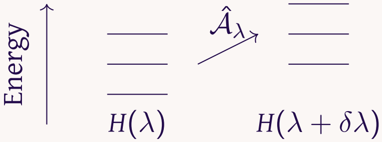
- Do higher moments of the AGP contain more information about chaos?
- Study open and driven systems.
- Connection between AGP variance, Lyapunov exponents, and operator growth?

Thank you!

Adiabatic Gauge Potential

Quantum AGP:

$$\hat{\mathcal{A}}_\lambda \left| n(\lambda) \right\rangle = i\hbar \, \partial_\lambda \left| n(\lambda) \right\rangle .$$



Elements of the AGP operator:

$$\langle n | \hat{\mathcal{A}}_\lambda(\mu) | m \rangle = \frac{i\omega_{mn}}{\omega_{mn}^2 + \mu^2} \langle n | \partial_\lambda H(\lambda) | m \rangle .$$

The regularizer $\mu \sim e^{-S}$ is added to remove singularities.

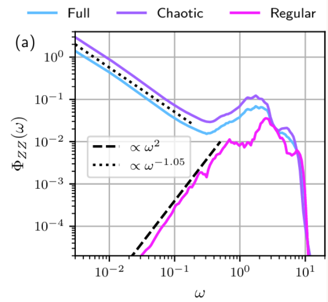
AGP norm:

$$||\hat{\mathcal{A}}_\lambda(\mu)||^2 = \frac{1}{\mathcal{D}} \sum_{m,n} | \langle m | \hat{\mathcal{A}}_\lambda(\mu) | n \rangle |^2 .$$

Adiabatic Gauge Potential

Spectral Function: $||\hat{\mathcal{A}}_\lambda(\mu)||^2 = \int_{-\infty}^{\infty} d\omega \, \frac{\omega^2}{(\omega^2+\mu^2)^2} \phi_\lambda(\omega).$

$$\phi_\lambda(\omega) \sim \begin{cases} 0, & \text{Integrable} \\ \omega^{-1+1/z}, & \text{Weak Chaos} \\ \text{constant}, & \text{Strong Chaos} \end{cases}$$



Lim, Matriko, Kim, Polkovnikov, & Flynn, 2024

Adiabatic Gauge Potential

Computing the AGP at a point in the phase space:

$$\mathcal{A}_\lambda(\mu) = \frac{1}{2}\int_{-\infty}^{\infty} dt \ \mathrm{sgn}(t) e^{-\mu|t|} \partial_\lambda H(t).$$

AGP on a trajectory:

$$\mathcal{A}_\lambda(t) = \mathcal{A}_\lambda(0) - \int_0^t d\tau \, \partial_\lambda H(\tau) + t\overline{\partial_\lambda H}.$$

Fermi–Pasta–Ulam–Tsingou (FPUT) Systems

Hamiltonian in normal mode space:

$$H = \sum_{k=1}^{N} \frac{P_k^2 + \omega_k^2 Q_k^2}{2} + \frac{\alpha}{3} \sum_{i,j,k=1}^{N} A_{ijk} Q_i Q_j Q_k + \frac{\beta}{4} \sum_{i,j,k,l=1}^{N} B_{ijkl} Q_i Q_j Q_k Q_l,$$

with frequencies $\omega = 2\sin\left(\frac{k\pi}{2(N+1)}\right)$, and

$$A_{ijk} = \frac{\omega_i \omega_j \omega_k}{\sqrt{2(N+1)}} \sum_{\pm} \left[\delta_{i\pm j\pm k,0} - \delta_{i\pm j\pm k,2(N+1)}\right],$$

$$B_{ijkl} = \frac{\omega_i \omega_j \omega_k \omega_l}{2(N+1)} \sum_{\pm} \left[\delta_{i\pm j\pm k\pm l,0} - \delta_{i\pm j\pm k\pm l,\pm 2(N+1)}\right].$$

Fermi–Pasta–Ulam–Tsingou (FPUT) Systems

Spectral entropy:

$$S = -\sum_{k=1}^{N} \varepsilon_k \ln \varepsilon_k, \ \varepsilon_k = \frac{E_k}{\sum_{k'=1}^{N} E_{k'}}.$$

Rescaled entropy:

$$\eta(t) = \frac{S(t) - S_{\max}}{S(0) - S_{\max}}.$$

Equilibrium expectation value:

$$\langle \eta \rangle = \frac{1 - \gamma}{S_{\max} - S(0)}.$$

Logistic Map

Map:

$$x_{n+1} = f(x_n) = rx_n(1 - x_n), \; 0 \leq r \leq 4.$$

Fixed points:

$$x = 0, \; x = 1 - 1/r.$$

The point $x = 0$ is stable for $r < 1$, while $x = 1 - 1/r$ is stable for $1 < r < 3$.

For $r > 3$, the limit cycle is formed by the fixed points of the kth iterate of the map, $f^{(k)}(x) = f(f(\ldots f(x)))$. If the 2^s-period orbit becomes stable at r_s, then:

$$\lim_{s \to \infty} r_s = 3.56994\ldots$$