The Metastable State of the FPUT Problem

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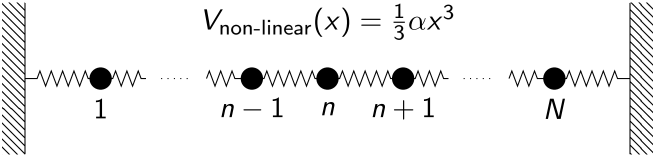
APS March Meeting

Recap

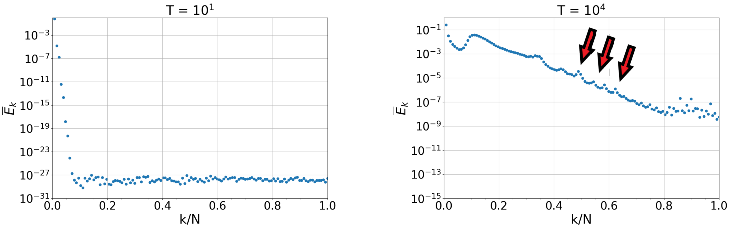


<u>Fourier Space</u>: $H = \sum_{k=1}^{N} \frac{P_k^2 + \omega_k^2 Q_k^2}{2} + \frac{\alpha}{3} \sum_{i,j,k} A_{ijk} Q_i Q_j Q_k.$

- Mode energies $E_k = \frac{P_k^2+\omega_k^2 Q_k^2}{2}$ are mixed due to the non-linearity.
- How is equipartition achieved? Does this reconcile with stat mech in the $N \to \infty$ limit?

The Metastable State

- Evolution of a 128 particle system initialized in the first mode:



Snapshots of the energy distribution at different times

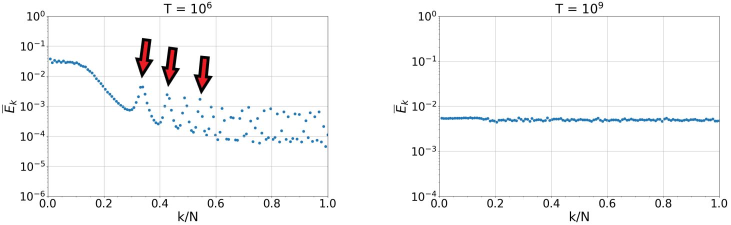
- Quick formation of a non-thermal (metastable) state.
- The system is trapped in the metastable state for a long time.

Benettin, G., et al (2007). The Fermi—Pasta—Ulam Problem and the Metastability Perspective.

https://doi.org/10.1007/978-3-540-72995-2_4

The Metastable State

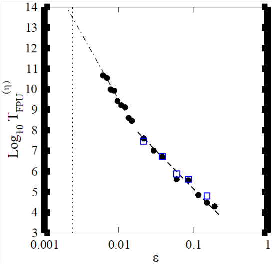
- Resonances lift up the spectrum slowly.



- Relaxation to equilibrium over a much longer time-scale.
- At even smaller energies, equilibrium cannot be achieved numerically due to long computation times.

The Metastable State

- Wave-turbulence theory predicts scaling of time to equilibrium: $t \sim \frac{1}{(E\alpha^2)^4}$.
- This trend is broken at smaller energy densities ($\epsilon = E/N$).
- Is there an energy threshold?

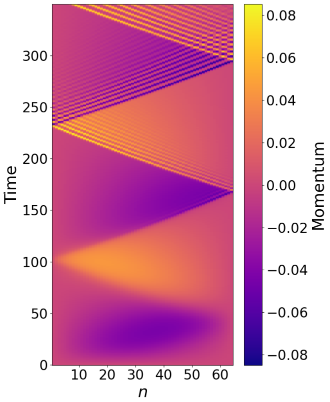


Time to equilibrium vs ϵ for a 32 particle system[1]. A cross-over is observed at $\epsilon \approx 0.01$.

[1] Danieli, C., et al. Intermittent many-body dynamics at equilibrium, Phys. Rev. E 2017; 95:060202.

The Metastable State and Solitons

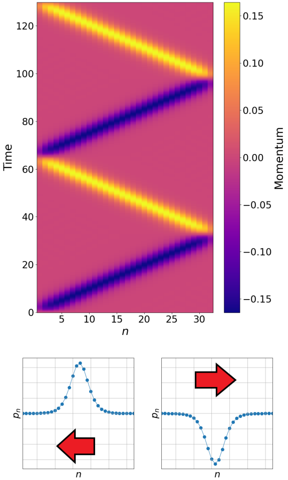
- The metastable state can be better understood by studying the mechanical momenta.
- A system initialized in the first mode starts out as a standing wave but quickly breaks up into quasi-solitons.
- The metastable state can be thought of as a perturbation to a state composed of quasi-solitons.



Formation of quasi-solitons from a standing wave

q-Breathers

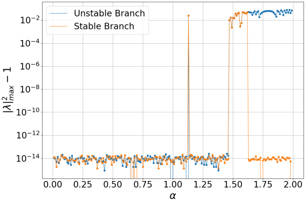
- Periodic orbits, or *q*-breathers[1] are composed of quasi-solitons.
- Normal modes continuously deform into *q*-breathers as non-linearity is added.
- Localized in the mode space around the seed mode.
- Stability of the *q*-breather affects the life-time of the metastable state.



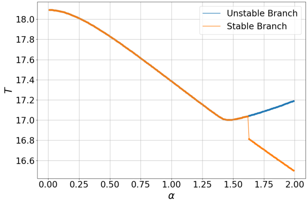
[1] Flach, S., et al. *q*-breathers in Fermi-Pasta-Ulam chains: Existence, localization, and stability. Phys. Rev. E 2006; 73:036618.

q-Breather Stability Analysis

- Linearly stable only if all Floquet multipliers lie on the unit circle.
- All q-breathers are stable for small non-linearities.
- Above a critical non-linearity, both stable and unstable breathers exist simultaneously.



Maximum Floquet multiplier deviation from the unit circle as a function of the non-linearity, α. A bifurcation is observed at $\alpha \approx 1.5$.

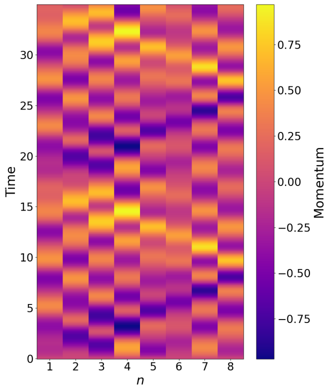


q-Breather period as a function of the non-linearity α.

q-Breather Stability Analysis

Unstable q-Breathers:

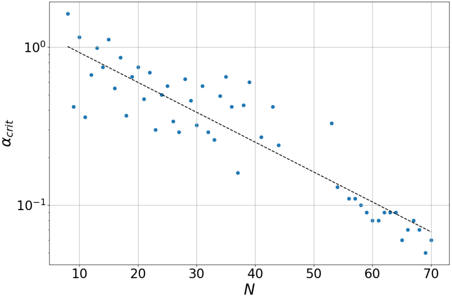
- Non-local in the mode space.
- Lose their soliton-like shape.
- Any nearby state quickly moves away from the breather towards equilibrium.



Evolution of an unstable breather

Stability Analysis of q-Breathers

- Non-linearity at which bifurcation happens goes down with increasing N at constant energy.



- Encounter instability sooner in larger systems.
- Expect any system above the criticality to eventually achieve equilibrium.

Discussion and Outlook

- The α-FPUT system goes through a long period of metastability on its way to equilibrium.
- Quasi solitons, or q-breathers, determine the dynamics of the metastable state.
- Stable and unstable breathers exist simultaneously above a certain non-linearity.
- Encounter instabilities sooner in larger systems.
- KAM regions?

References I

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