Understanding Chaos through Physical Observables

Nachiket Karve, Nathan Rose, Hyeongjin Kim, David Campbell, and Anatoli Polkovnikov

Department of Physics, Boston University, Boston, Massachusetts 02215, USA

Introduction

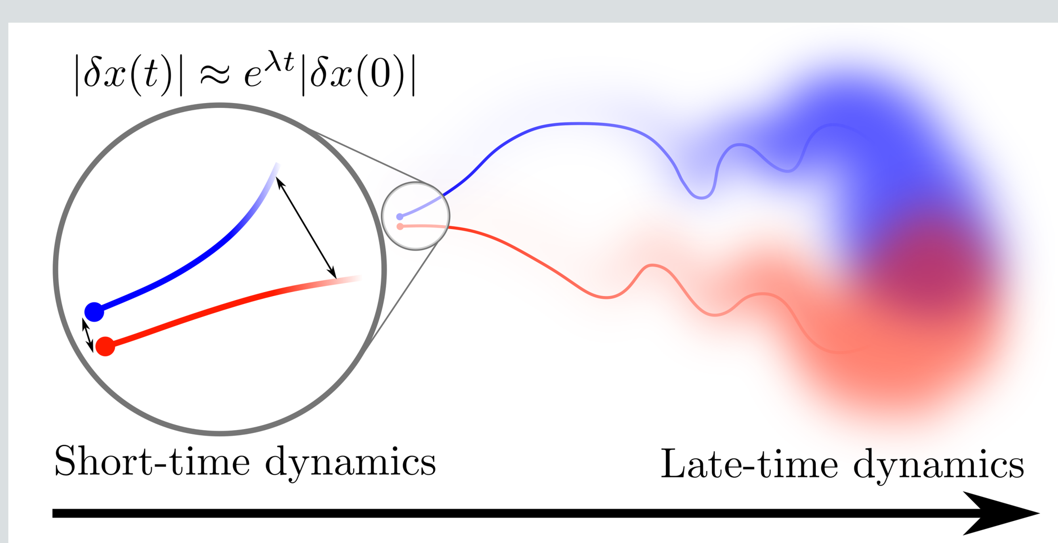
- The most widely accepted definition of chaos in classical systems is the sensitivity to initial conditions. This sensitivity is quantified by the Lyapunov exponent, given by

$$\lambda = \lim_{t\to\infty} \lim_{|\delta x(0)|\to 0} \frac{1}{t} \ln \frac{|\delta x(t)|}{|\delta x(0)|}. \qquad (1)$$

- However, it is almost impossible to compute or measure the Lyapunov exponent in real-world systems. Such systems typically contain over 10^{23} degrees of freedom, and the exact microscopic laws governing their dynamics are often unknown.

The difference between calm and stormy weather is lost at the microscopic level. If one is able to compute the Lyapunov exponent, it would be identical for both situations.

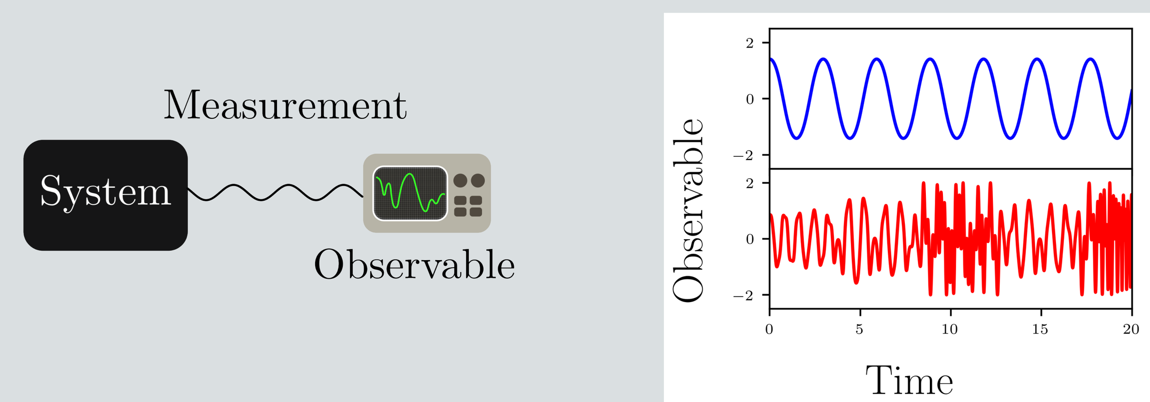
- The Lyapunov exponent is a short-time diagnostic of chaos. It is unable to capture long-time behavior.



Late-time behavior is more appropriately described through statistical properties, rather than individual trajectories.

<u>Solution:</u> Define chaos (classical or quantum) solely through quantities that can be measured – *physical observables.*

Correlation Functions and Noise Spectra



A typical laboratory experiment. A regular system produces smooth and periodic signals, while a chaotic system produces noisy signals.

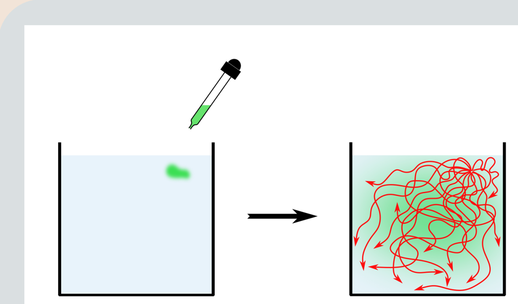
- An observable in a chaotic system becomes more and more uncorrelated with itself over time. That is, the autocorrelation function $C(t) = \langle O(t)O(0)\rangle - \langle O\rangle^2$ of an observable O decays to zero: $\lim_{t\to\infty} C(t) = 0$.
- Three different regimes of interest based on long-time behavior / low-frequency noise:

Regular behavior	$C(t) \not\to 0$
Slow relaxation	$C(t) > \mathcal{O}(1/t)$
Fast relaxation	$C(t) < \mathcal{O}(1/t)$

Slowly relaxing systems exhibit the highest sensitivity to external perturbations!

Sensitivity to Initial Conditions

- Decaying correlations are a property of mixing systems. Such systems also exhibit sensitivity to initial conditions.
- Can quantify this sensitivity through diffusion [1]:



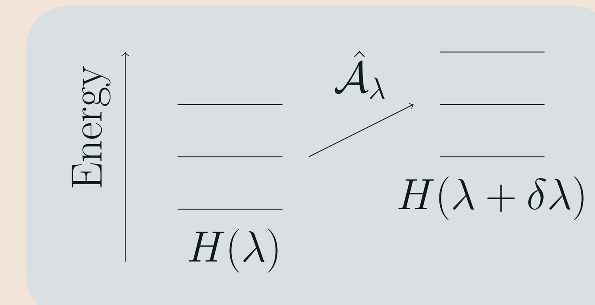
Phase space of a mixing system is akin to ink diffusing in water. Any nearby initial conditions eventually lead to completely different final states.

Regular systems $\Longrightarrow$ No diffusion

Slow relaxation $\Longrightarrow$ Super-diffusion

Fast relaxation $\Longrightarrow$ Normal diffusion

Sensitivity to Adiabatic Deformations



- Consider adiabatically varying a parameter λ in a quantum system. The change in its energy levels is captured by the adiabatic gauge potential (AGP) $\hat{\mathcal{A}}_\lambda$.
- The AGP norm can be used to quantify the amount of chaos in a system [2]:

Regular systems $\Longrightarrow$ Finite $||\mathcal{A}_\lambda||^2$

Slow relaxation $\Longrightarrow$ Super-linearly diverging $||\mathcal{A}_\lambda||^2$

Fast relaxation $\Longrightarrow$ Linearly diverging $||\mathcal{A}_\lambda||^2$

Sensitivity to Quenches

- Consider quenching a parameter λ in a system (either classical or quantum):

$$\lambda(t) = \lambda_0 + \epsilon\Theta(t). \qquad (2)$$

- The linear response $\frac{d}{d\epsilon}\langle O\rangle$ [3] of an operator O captures the amount of chaos in the system:

Regular systems $\Longrightarrow$ Finite response

Slow relaxation $\Longrightarrow$ Divergent response

Fast relaxation $\Longrightarrow$ Finite response

Conclusions

- Chaos can be defined through the long-time behavior of physical observables, both in classical and quantum systems.
- **Slow relaxation $\Longrightarrow$ the highest sensitivity to external perturbations $\Longrightarrow$ strong chaos.**

[1] N. Karve, N. Rose, and D. Campbell. Diffusion as a signature of chaos, 2025.

[2] H. Kim, C. Lim, K. Matirko, A. Polkovnikov, and M. O. Flynn. Defining classical and quantum chaos through adiabatic transformations, 2025.

[3] D. Ruelle. A review of linear response theory for general differentiable dynamical systems. *Nonlinearity*, 22(4):855–870, March 2009.

See more: